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1997 J. Phys. A: Math. Gen. 30 3983

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J. Phys. A: Math. Gen. **30** (1997) 3983–3990. Printed in the UK PII: S0305-4470(97)75046-7

Representations of generalized oscillator algebra

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Received 29 May 1996, in final form 11 March 1997

Abstract. The representations of the oscillator algebra introduced by Brzeziński *et al* (Brzeziński T, Egusquiza J L and Macfarlane A J 1993 *Phys. Lett.* **311B** 202) are classified.

1. Introduction

In this paper we present an exhausitive discussion of the representations of the deformed oscillator algebra introduced in [1]. This algebra was obtained as a q-deformation of the Heisenberg algebra endowed with the Klein reflection operator (actually, it depends on two parameters, q and α; the case $q = 1$ will be referred to as the undeformed one, irrespective of the value of α). The undeformed version of the algebra appeared for the first time in [2, 3]; it can be viewed [4] as a special case of generalized oscillator algebras depending on arbitrary function [5] (cf also [6]) once the Klein operator is realized in terms of the creation–annihilation operators [7].

Plyushchay [4] has noted that such an algebra can be applied to the construction of the representation of the universal covering of the $SL(2, R)$ group and, consequently, to the construction of the free field equations in $2 + 1$ dimensions; it can also be used to bosonize the supersymmetric quantum mechanics.

The generalized version of the above algebra appeared to provide the algebraic structure underlying the complete integrability of quantum-mechanical N-body Calogero model [8–10] and turned out to be useful in establishing the link between the Knizhnik–Zamolodchikov equations and the Calogero model [11]. Similar structures appear in Turbiner's paper [12] that also concern exact intergrability of the Calogero model. Subsequently, further applications have been found including, for example, new realizations of the Virasoro algebra and a new class of the W_∞-type algebras containing higher-spin currents together with the Virasoro generators [13].

The q-deformation of the simplest version of the Calogero–Vasiliev algebra was introduced in [1] (see also [14]).

In most of the above papers the Fock-type representations of the (deformed) Calogero–Vasiliev algebra were studied (see, for example, [15] where Schwinger's method is used). However, it is known [16] that the q-deformed oscillator algebra posses some exotic representations which disappear in the limit $q \to 1$. Therefore, it is expected that the same phenomenon occurs for the algebra considered by Brzeziński *et al*. Following the methods used in [16] we show that this is indeed the case, and provide the complete classification of its representations. In particular, we show that, apart from the Fock representations there

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exist four further types which, however, (with one exception) do not survive the $q \to 1$ limit.

2. Construction of representations

The algebra under consideration reads

$$\begin{aligned} &aa^+ - qa^+a = q^{-N}(1+2\alpha K) \\ &[N,a] = -a \qquad [N,a^+] = a^+ \\ &\{K,a\} = 0 \qquad \{K,a^+\} = 0 \\ &[N,K] = 0 \\ &N^+ = N \qquad K^+ = K \end{aligned} \tag{1}$$

where $q \in \mathbb{R}_+$, $\alpha \in \mathbb{R} - \{0\}$. This algebra possesses the following Casimir operators:

$$C_1 = K^2 \qquad C_2 = K\mathrm{e}^{\mathrm{i}\pi N} \qquad C_3 = \mathrm{e}^{2\mathrm{i}\pi N}. \tag{2}$$

Obviously, they are not independent:

$$C_1 C_3 = C_2^2. \tag{3}$$

We will be looking for irreducible representations. Let γ be an eigenvalue of C_2 corresponding to a given representation; then

$$K = \gamma \mathrm{e}^{-\mathrm{i}\pi N}. \tag{4}$$

Let Ψ_0 be a common eigenvector of N and K:

$$\begin{aligned} &N\Psi_0 = \nu_0 \Psi_0 \\ &K\Psi_0 = \gamma \mathrm{e}^{-\mathrm{i}\pi\nu_0}\Psi_0. \end{aligned} \tag{5}$$

Due to the commutativity of a^+a and aa^+ with N and K we may assume that

$$\begin{aligned} &a^+a\Psi_0 = \lambda_0\Psi_0 \\ &aa^+\Psi_0 = \mu_0\Psi_0 \end{aligned} \tag{6}$$

and $(\Psi_0, \Psi_0) = 1$. It is easy to see that the vectors Φ_n defined by

$$\Phi_n = \begin{cases} \left(a^+\right)^n \Psi_0 & \text{for } n \geqslant 0 \\ a^{-n}\Psi_0 & \text{for } n < 0 \end{cases}$$

are eigenvectors of a^+a and aa^+:

$$\begin{aligned} &a^+a\Phi_n = \lambda_n\Phi_n \\ &aa^+\Phi_n = \mu_n\Phi_n. \end{aligned} \tag{7}$$

Now, let us define the following vectors:

$$\Psi_n = \begin{cases} \dfrac{1}{\sqrt{\prod_{k=1}^{n}\lambda_k}}\left(a^+\right)^n\Psi_0 & \text{for } n \geqslant 0 \\[2ex] \dfrac{1}{\sqrt{\prod_{k=1}^{-n}\lambda_{n+k}}}a^{-n}\Psi_0 & \text{for } n < 0. \end{cases} \tag{8}$$

They are orthogonal (as eigenvectors of N corresponding to different eigenvalues) and normalized. The action of the basic operators is given by

$$
\begin{aligned}
a^+\Psi_n &= \sqrt{\lambda_{n+1}}\Psi_{n+1} \\
a\Psi_n &= \sqrt{\lambda_n}\Psi_{n-1} \\
N\Psi_n &= (\nu_0 + n)\Psi_n \\
K\Psi &= \frac{(-1)^n}{2\alpha} B\Psi_n
\end{aligned}
\tag{9}
$$

where, for later convenience, we have defined

$$B = 2\alpha\gamma \mathrm{e}^{-\mathrm{i}\pi\nu_0} \in \mathbb{R}. \tag{10}$$

The only additional condition we have to take into account is that λ_n and μ_n, being eigenvalues of non-negative operators, should be non-negative. Using the basic commutation rules applied to Ψ_n we obtain

$$\mu_n - q\lambda_n = -q^{-(n+\nu_0)}\left(1 + 2\alpha\gamma \mathrm{e}^{-\mathrm{i}\pi(n+\nu_0)}\right). \tag{11}$$

However, $a\left(a^+a\right)\Psi_n = \left(aa^+\right)a\Psi_n$ which gives

$$\lambda_n = \mu_{n-1}. \tag{12}$$

Equations (11) and (12) imply the following recurrence relation:

$$\lambda_{n+1} = q\lambda_n + q^{-\nu_0-n}\left(1 + (-1)^n B\right) \tag{13}$$

which can be explicitly solved to yield

$$\lambda_n = \lambda_0 q^n + q^{-\nu_0}\left(\frac{q^n - q^{-n}}{q - q^{-1}} + B\frac{q^n - (-1)^n q^{-n}}{q + q^{-1}}\right). \tag{14}$$

Non-negativity of λ_n implies

$$\lambda_0 q^{\nu_0} + \frac{1}{q - q^{-1}} + \frac{B}{q + q^{-1}} \geqslant q^{-4k}\left(\frac{1}{q - q^{-1}} + \frac{B}{q + q^{-1}}\right) \tag{15a}$$

$$\lambda_0 q^{\nu_0} + \frac{1}{q - q^{-1}} + \frac{B}{q + q^{-1}} \geqslant q^{-(4k+2)}\left(\frac{1}{q - q^{-1}} - \frac{B}{q + q^{-1}}\right). \tag{15b}$$

We have now to distinguish several cases.

(i) Assume $q > 1$. Then at least one of the numbers

$$\frac{1}{q - q^{-1}} \pm \frac{B}{q + q^{-1}}$$

is positive. Therefore, there exists n_0 such that for even and/or odd $n < n_0$, $\lambda_n < 0$, which implies $a\Psi_n = 0$ for some $n \leqslant n_0$. After possible renumbering we may assume

$$a\Psi_0 = 0 \qquad \lambda_0 = 0. \tag{16}$$

Therefore, the representation is spanned by the vectors Ψ_n, $n \geqslant 0$, and λ_n are given by

$$\lambda_n = q^{-\nu_0+n}\left(\frac{1 - q^{-2n}}{q - q^{-1}} + B\frac{1 - (-1)^n q^{-2n}}{q + q^{-1}}\right). \tag{17}$$

The condition $\lambda \geqslant 0$ gives the following restriction on the possible values of B:

$$B \geqslant -1. \tag{18}$$

However, $B = -1$ must be considered separately. In this case $\lambda_1 = 0$, i.e. $\mu_0 = 0$ which, together with $\lambda_0 = 0$ and irreducibility implies

$$a = a^+ = 0 \qquad N = \nu_0 \qquad K = -\frac{1}{2\alpha}. \tag{19}$$

This representation is one dimensional. For $B > -1$ the representation is spanned by the vectors $\{\Psi_n\}_{n=0}^{\infty}$. We shall call it the Fock representation for obvious reasons. It is given by equations (9) and (17) with $n \geqslant 0$.

(ii) $q < 1$ and one (and only one) of the values

$$\frac{1}{q - q^{-1}} \pm \frac{B}{q + q^{-1}}$$

is positive. In this case there exists n_0 such that for $n > n_0$ λ_n is negative for even or odd n. This implies $a^+\Psi_n = 0$ for some $n \geqslant n_0$. After possible renumbering we get

$$a^+\Psi_0 = 0. \tag{20}$$

In order to find the restrictions on possible values of B we note that the condition (20) implies μ_0, i.e. $\lambda_1 = 0$ or

$$\lambda_0 = -q^{-\nu_0 - 1}(1 + B) \tag{21}$$

which gives $B \leqslant -1$. For $B = -1$ we obtain the one-dimensional representation (19). If $B < -1$ we have to consider the restrictions on B following from the formula

$$\lambda_n = q^{n-\nu_0}\left(-q^{-1}(1 + B) + \frac{1 - q^{-2n}}{q - q^{-1}} + B\frac{1 - (-1)^n q^{-2n}}{q + q^{-1}}\right). \tag{22}$$

The condition $\lambda_n \geqslant 0$ implies

$$B \leqslant \frac{q + q^{-1}}{q - q^{-1}}.$$

For

$$B < \frac{q + q^{-1}}{q - q^{-1}} \tag{23}$$

we have $\lambda_n > 0$ and the representation is given by equations (9) and (22) with $n \leqslant 0$. We call this representation the anti-Fock one. For

$$B = \frac{q + q^{-1}}{q - q^{-1}} \tag{24}$$

all λ_n with odd n are zero. Therefore the representation is two dimensional and given by

$$\begin{aligned} &a\Psi_0 = \sqrt{\frac{2q^{-\nu_0}}{q^{-1} - q}}\,\Psi_{-1} \qquad a^+\Psi_0 = 0 \\ &a^+\Psi_{-1} = \sqrt{\frac{2q^{-\nu_0}}{q^{-1} - q}}\,\Psi_0 \qquad a\Psi_{-1} = 0 \\ &N\Psi_0 = \nu_0\Psi_0 \qquad N\Psi_{-1} = (\nu_0 - 1)\Psi_{-1} \\ &K\Psi_0 = \frac{q + q^{-1}}{2\alpha\left(q - q^{-1}\right)} \qquad K\Psi_{-1} = \frac{q + q^{-1}}{2\alpha\left(q - q^{-1}\right)}\Psi_{-1}. \end{aligned} \tag{25}$$

(iii) $q < 1$ and both values

$$\frac{1}{q - q^{-1}} \pm \frac{B}{q + q^{-1}}$$

are non-positive (at least one must be strictly negative). There are now the following possibilities:

(a)
$$\lambda_0 q^{\nu_0} + \frac{1}{q - q^{-1}} + \frac{B}{q + q^{-1}} < 0. \tag{26}$$

Then there exists n_0 such that $\lambda_n < 0$ for $n < n_0$, n even or odd. Therefore the representation is given by equations (9), (17); it is a Fock one. To provide $\lambda_n \geqslant 0$ for $n \geqslant 0$ we have to restrict B to lie in the interval

$$-1 \leqslant B < -\frac{q + q^{-1}}{q - q^{-1}}. \tag{27}$$

For $B = -1$ we again get a one-dimensional representation (19).

(b)
$$\lambda_0 q^{\nu_0} + \frac{1}{q - q^{-1}} + \frac{B}{q + q^{-1}} > 0. \tag{28}$$

Equation (28) implies $\lambda_n > 0$ for all $n \in \mathbb{Z}$. The representation is given by equations (9), (14) with $n \in \mathbb{Z}$.

(c)
$$\lambda_0 q^{\nu_0} + \frac{1}{q - q^{-1}} + \frac{B}{q + q^{-1}} = 0. \tag{29}$$

If

$$|B| < -\frac{q + q^{-1}}{q - q^{-1}}$$

all $\lambda_n > 0$ and the representation has the same form as in (b). For

$$B = -\frac{q + q^{-1}}{q - q^{-1}}$$

all λ_n with n even are vanishing; therefore, the representation is two dimensional and is given by the formulae

$$\begin{aligned}
&a^+\Psi_0 = \sqrt{\frac{2q^{-\nu_0-1}}{q^{-1} - q}}\,\Psi_1 \qquad a\Psi_0 = 0 \\
&a^+\Psi_1 = 0 \qquad a\Psi_1 = \sqrt{\frac{2q^{-\nu_0-1}}{q^{-1} - q}}\,\Psi_0 \\
&N\Psi_0 = \nu_0\Psi_0 \qquad N\Psi_1 = (\nu_0 + 1)\Psi_1 \\
&K\Psi_0 = -\frac{q + q^{-1}}{2\alpha\left(q - q^{-1}\right)}\Psi_0 \qquad K\Psi_1 = \frac{q + q^{-1}}{2\alpha\left(q - q^{-1}\right)}\Psi_1.
\end{aligned} \tag{30}$$

For

$$B = \frac{q + q^{-1}}{q - q^{-1}}$$

all λ_n with n odd vanish. The representation is two dimensional and is given by equations (25).

3. Discussion

Let us summarize the results obtained in section 2. For $q > 1$ there are two possibilities:

(a) if $B > -1$ the spectrum of N is bounded from below, and we have Fock representation which is irreducible and determined by the choice of ν_0 and B; different choices correspond to inequivalent representations;

(b) if $B = -1$ we get a one-dimensional irreducible representation labelled by the values of ν_0; again different values of ν_0 correspond to inequivalent representations.

The case $q < 1$ is more involved. The following possibilities have to be distinguished:

(a) for

$$B < \frac{q + q^{-1}}{q - q^{-1}}$$

we get irreducible anti-Fock representation; the representations are labelled by pairs (ν_0, B) and different choices correspond to inequivalent representations;

(b) for

$$B = \frac{q + q^{-1}}{q - q^{-1}}$$

one obtains two-dimensional representations parametrized by ν_0; different values of ν_0 correspond to inequivalent representations;

(c) for $B = -1$ we again obtain a one-dimensional representation parametrized by ν_0; it has the same form as for $q > 1$;

(d) for

$$-1 < B < -\frac{q + q^{-1}}{q - q^{-1}}$$

the representations are the Fock ones parametrized by ν_0 and B; for different values of these parameters we obtain inequivalent representations;

(e) for

$$B = -\frac{q + q^{-1}}{q - q^{-1}}$$

the representations, parametrized by ν_0, are two dimensional and mutually inequivalent;

(f) finally, there exists a set of infinite-dimensional representations for which the spectrum of N extends infinitely in both directions. They correspond to

$$|B| < -\frac{q + q^{-1}}{q - q^{-1}}$$

and

$$\lambda_0 q^{\nu_0} + \frac{1}{q - q^{-1}} + \frac{B}{q + q^{-1}} \geqslant 0$$

or

$$|B| = -\frac{q + q^{-1}}{q - q^{-1}}$$

and

$$\lambda_0 q^{\nu_0} + \frac{1}{q-q^{-1}} + \frac{B}{q+q^{-1}} > 0.$$

It is easy to see that two such representations, labelled by (ν_0, B, λ_0) and (ν_0', B', λ_0'), are equivalent iff

$$\nu_0' = \nu_0 + n \qquad B' = (-1)^n B \qquad \lambda_0' = \lambda_0 q^n + q^{-\nu_0}\left(\frac{q^n - q^{-n}}{q-q^{-1}} + B\frac{q^n - (-1)^n q^{-n}}{q+q^{-1}}\right)$$

for some integer n.

Let us now consider the limits $q \to 1$ or $B \to 0$. It is easy to see that only the one-dimensional and Fock representations survive the limit $q \to 1$. On the other hand, the $B \to 0$ limit coincides with the results obtained in [16]. Finally, the limit $q \to 1$, $B \to 0$ leaves only the Fock representation as it should be. The results obtained are summarized in table 1.

Table 1.

Type of representation	q	Restrictions on B	Restrictions on λ_0 and ν_0	$q \to 1$ limit	$\alpha \to 1$ limit
One dimensional	Arbitrary	$B = -1$	$\lambda_0 = 0$, ν_0 arbitrary	Exists	Does not exist
Two dimensional	$q < 1$	$B = \frac{q+q^{-1}}{q-q^{-1}}$	$\lambda_0 = \frac{2q^{-\nu_0}}{q^{-1}-q}$, ν_0 arbitrary	Does not exist	Does not exist
		$B = -\frac{q+q^{-1}}{q-q^{-1}}$	$\lambda_0 = 0$, ν_0 arbitrary	Does not exist	Does not exist
Fock	$q > 1$	$B > -1$	$\lambda_0 = 0$, ν_0 arbitrary	Exists	Exists
	$q < 1$	$-\frac{q+q^{-1}}{q-q^{-1}} > B > -1$	$\lambda_0 = 0$, ν_0 arbitrary	Exists	Exists
Anti-Fock	$q < 1$	$B < \frac{q+q^{-1}}{q-q^{-1}}$	$\lambda_0 = -q^{\nu_0 - 1}(1+B)$, ν_0 arbitrary	Does not exist	Does not exist
Unbounded in both directions	$q < 1$	$\|B\| < -\frac{q+q^{-1}}{q-q^{-1}}$	$\lambda_0 q^{\nu_0} + \frac{1}{q-q^{-1}} + \frac{B}{q+q^{-1}} \geqslant 0$	Does not exist	Exists
		$\|B\| = -\frac{q+q^{-1}}{q-q^{-1}}$	$\lambda_0 q^{\nu_0} + \frac{1}{q-q^{-1}} + \frac{B}{q+q^{-1}} > 0$	Does not exist	Does not exist

Acknowledgments

PK and MM acknowledge financial support from the KBN under grant No 2P30221706p02. PM acknowledges financial support from the University of Łódź under grant No 505/444.

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